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INTEGRAL INEQUALITIES VIA STRONGLY BETA-CONVEX FUNCTIONS

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Abstract. We introduce and study a new class of convex functions which is called strongly beta-convex functions. We derive some new Hermite-Hadamard inequalities. Some special cases are also considered. The techniques of this paper may be starting point for further research in this dynamic field.

Keywords: Strongly beta-convex functions, Beta-convex functions, Hermite-Hadamard type inequality.

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1. Introduction

Inequalities played a fundamental role in the development of almost all the fields of pure and applied sciences. Inequalities present very active and fascinating field of research. In recent years, a wide class of integral inequalities are being derived via different concepts of convexity. A significant class of convex functions is strongly convex function introduced by Polyak [20]. He used the strongly convex function for proving the convergence of a gradient type algorithm for minimizing a function. They also play an important role in optimization theory, mathematical economics, variational inequalities and other branches of pure and applied mathematics, see [2, 5, 10, 12, 13, 15-26]. Merentes and Nikodem [10] obtained the Hermite-Hadamard inequality for strongly convex functions. Another class of convex functions is considered by Tunc et. al [24], which is called beta-convex function.

Motivated and inspired by the ongoing research in this dynamic field, we introduce and investigate a new class of convex functions, called strongly beta-convex functions. We discuss some properties of strongly beta-convex functions. Some new Hermite-Hadamard integral inequalities involving beta function are obtained. Some special cases are also discussed. Results obtained in this paper can be viewed as significant refinement and improvement of the known results.

We now recall the basic concepts and results, which are needed to obtain the main results.

Definition 1.1 [19]. A set $I \subseteq \mathbb{R}$ is said to be a convex set, if

$$(1-t)x+ty \in I, \qquad \forall x, y \in I, t \in [0,1].$$

We now introduce a new concept of strongly (k,h_1,h_2)-convex function, which includes the main defintion of Crestescu et al [4].

Definition 1.2 A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be strongly (k,h_1,h_2)-convex function with respect to non-negative functions $k,\ h_1,\ h_2$ with modulus $c > 0$, if

$$f\left(k(1-t)x + k(t)y\right) \leq h_1(t) f(x) + h_2(t) f(y) - ct(1-t)(x-y)^2, \quad \forall x, y \in I, t \in (0,1). \tag{1}$$

We note that if $c = 0$, then Definition 1.2 reduces the concept of (k,h_1,h_2)-convexity defined in [4].

If $k(t) = t, h_1(t) = (1-t)^p t^q$ and $h_2(t) = t^p (1-t)^q$, then Definition 1.2 reduces to strongly beta-convex functions.

Definition 1.3 A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be strongly beta-convex function with modulus $c > 0$, where $p,q \geq -1$, if

$$f\left((1-t)x + ty\right) \leq (1-t)^p t^q f(x) + t^p (1-t)^q f(y) - ct(1-t)(x-y)^2, \quad \forall x, y \in I, t \in (0,1). \tag{2}$$

If (2) is assumed for $t = \frac{1}{2}$, then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2^{p+q}} - \frac{c}{4}(x-y)^2, \quad \forall x, y \in I,$$

which is called Jensen strongly beta-convex function.

If $c = 0$, then strongly beta-convex functions are exactly the generalized beta-convex functions studied by Noor [14]. This notion unifies and generalizes the known classes of strongly convex functions, Godunova-Levin strongly convex functions, strongly P-functions, strongly tgs-convex functions, strongly s-convex functions, Godunova-Levin strongly s-convex functions and generalized strongly MT-convex functions, which are obtained by considering

$$\left(p,q\right) = \left\{(1,0), (-1,0), (0,0), (1,1), (s,0), (-s,0), (\frac{1}{2}, -\frac{1}{2})\right\}, \text{ respectively.}$$

Definition 1.4 A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be strongly $\log$-beta-convex function with respect to non-negative functions $k,\ h_1,\ h_2$ with on I, if

$$f\left(k(1-t)x+k(t)y\right)\leq [f(x)]^{h_1(t)}[f(y)]^{h_2(t)}-ct(1-t)(x-y)^2, \quad \forall x,y\in I, t\in(0,1).$$

For $t=\frac{1}{2}$, we have

$$f\left(k(\frac{1}{2})(x+y)\right)\leq [f(x)]^{h_1(\frac{1}{2})}[f(y)]^{h_2(\frac{1}{2})}-\frac{c}{4}(x-y)^2, \qquad \forall x,y\in I,$$

which is called Jensen type strongly log -beta-convex function.

If $k(t)=t, h_1(t)=(1-t)^p t^q$ and $h_2(t)=t^p(1-t)^q$, then the function f is said to be strongly log -beta-convex functions with modulus $c>0$, that is,

$$\begin{aligned} f\left((1-t)x+ty\right) &\leq [f(x)]^{(1-t)^p t^q}[f(y)]^{t^p(1-t)^q}-ct(1-t)(x-y)^2 \\ &\leq (1-t)^p t^q f(x)+t^p(1-t)^q f(y)-ct(1-t)(x-y)^2. \end{aligned}$$

This shows that, strongly log -beta-convex function implies strongly beta-convex function, but the converse is not true.

If we take $(p,q)=(1,0)$, then Definition 1.4, reduces to the definition of strongly log -convex functions introduced by Sarikaya et. al [22].

We recall the following special function which is known as Beta function.

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0,$$

where $\Gamma(\cdot)$ is a Gamma function.

2. Main Results

Lemma 2.1.

i). Let function $f: I=[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a strongly beta-convex function with modulus c. If $t^p(1-t)^q\leq t$, then the function $g(x)=f(x)-c(x)^2$ is beta-convex.

ii). Let function $f: I=[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a strongly beta-mid-convex with modulus c. If $\frac{1}{2^{p+q}}\leq\frac{1}{2}$, then the function $g(x)=f(x)-c(x)^2$ is beta-mid-convex function.

Proof.

i) Assume that f is strongly beta-convex with modulus c. Using properties of the inner product and assumption $t^p(1-t)^q\leq t$, we have

$$g((1-t)x+ty)= f((1-t)x+ty)-c((1-t)x+ty)^2 \leq$$
$$\leq (1-t)^p t^q f(x)+t^p(1-t)^q f(y)-ct(1-t)(x-y)^2 -$$
$$-c((1-t)x+ty)^2 =$$
$$=(1-t)^p t^q f(x)+t^p(1-t)^q f(y)-c(t(1-t)\left[(x)^2-2xy+(y)^2\right]+$$
$$+(1-t)^2(x)^2+2t(1-t)xy+t^2(y)^2)=$$
$$=(1-t)^p t^q f(x)+t^p(1-t)^q f(y)-c(1-t)(x)^2-ct(y)^2 \leq$$
$$\leq (1-t)^p t^q f(x)+t^p(1-t)^q f(y)-c(1-t)^p t^q (x)^2$$
$$-ct^p(1-t)^q(y)^2 = (1-t)^p t^q g(x)+t^p(1-t)^q g(y)$$

which gives that g is beta-convex function. This shows that f is strongly beta-convex with modulus c.
ii). Let f be strongly beta-mid-convex with modulus c and assumption $\frac{1}{2^{p+q}} \leq \frac{1}{2}$. Then

$$g\left(\frac{x+y}{2}\right)= f\left(\frac{x+y}{2}\right)-c(\frac{x+y}{2})^2 \leq$$
$$\leq \frac{f(x)+f(y)}{2^{p+q}}-\frac{c}{4}(x-y)^2-\frac{c}{4}(x+y)^2 \leq$$
$$\leq \frac{f(x)+f(y)}{2^{p+q}}-\frac{c}{4}[2(x)^2+2(y)^2] \leq$$
$$\leq \frac{[f(x)-(x)^2]+[f(y)-(y)^2]}{2^{p+q}} = \frac{g(x)+g(y)}{2^{p+q}}$$

which implies that g is beta-mid-convex function.

Now we drive Hermite-Hadamard inequality for strongly beta convex functions.

Theorem 2.1. Let $f: I=[a,b]\subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be strongly beta-convex function with modulus $c>0$. If $f \in L[a,b]$, then

$$2^{p+q-1}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(a-b)^2\right] \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$
$$\leq \beta(p+1,q+1)[f(a)+f(b)]-\frac{c}{6}(a-b)^2.$$

Proof. Let f be strongly beta-convex function. Then taking $x=(1-t)a+tb$ and $y=ta+(1-t)b$ in (2), we have

$$f\left(\frac{a+b}{2}\right)\le \frac{1}{2^{p+q}}\left[f\left((1-t)a+tb\right)+f\left(ta+(1-t)b\right)\right]-$$

$$-\frac{c(1-2t)^2}{4}(a-b)^2=$$

$$=\frac{1}{2^{p+q}}\left[\int_0^1 f((1-t)a+tb)\mathrm{d}t+\int_0^1 f(ta+(1-t)b)\mathrm{d}t\right]-$$

$$-\frac{c}{4}(a-b)^2\int_0^1(1-2t)^2\mathrm{d}t$$

This implies

$$2^{p+q-1}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(a-b)^2\right]\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

Now consider

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x=\int_0^1 f((1-t)a+tb)\mathrm{d}t\le$$

$$\le f(a)\int_0^1(1-t)^p t^q\mathrm{d}t+f(b)\int_0^1 t^p(1-t)^q\mathrm{d}t-$$

$$-c(a-b)^2\int_0^1 t(1-t)\mathrm{d}t=$$

$$=\beta(p+1,q+1)[f(a)+f(b)]-\frac{c}{6}(a-b)^2,$$

which is the required result.

Note that, if $p,q=1$ and $c=0$ in Theorem 2.1, then it becomes a particular case of right hand side of the Hermite Hadamard inequality (3.1), see [4].

Theorem 2.2. Let $f,g:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be strongly beta convex functions with modulus $c>0$. If $f,g\in L[a,b],$ then

$$\frac{1}{b-a}\int_a^b f(x)g(a+b-x)\mathrm{d}x\le$$

$$\le\beta(p+q+1,p+q+1)M(a,b)+\beta(2p+1,2q+1)N(a,b)-$$

$$-\beta(p+2,q+2)c(a-b)^2S(a,b)+\frac{c^2}{30}(a-b)^4,$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b) \tag{4}$$

$$N(a,b)=f(a)g(b)+f(b)g(a) \tag{5}$$

$$S(a,b) = f(a) + f(b) + g(a) + g(b). \qquad (6)$$

Proof. Let f, g be strongly beta-convex functions with modulus $c > 0$. Then

$$\frac{1}{b-a}\int_a^b f(x)g(a+b-x)\mathrm{d}x =$$

$$= \int_0^1 f((1-t)a+tb)g(ta+(1-t)b)\mathrm{d}t \le$$

$$\le \int_0^1 \begin{bmatrix} (1-t)^p t^q f(a) + t^p(1-t)^q f(b) \\ -ct(1-t)(a-b)^2 \end{bmatrix} \begin{bmatrix} t^p(1-t)^q g(a) + (1-t)^p t^q g(b) \\ -ct(1-t)(a-b)^2 \end{bmatrix} =$$

$$= [f(a)g(a) + f(b)g(b)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t +$$

$$+[f(a)g(b) + f(b)g(a)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t -$$

$$-c(a-b)^2[f(a)+f(b)+g(a)+g(b)]\int_0^1 t^{p+1}(1-t)^{q+1}\mathrm{d}t.$$

$$+c^2(a-b)^4\int_0^1 t^2(1-t)^2\mathrm{d}t =$$

$$= \beta(p+q+1, p+q+1)[f(a)g(a)+f(b)g(b)] +$$
$$+\beta(2p+1, 2q+1)[f(a)g(b)+f(b)g(a)] -$$
$$-\beta(p+2, q+2)c(a-b)^2[f(a)+f(b)+g(a)+g(b)] + \frac{c^2}{30}(a-b)^4 =$$

$$= \beta(p+q+1, p+q+1)M(a,b) + \beta(2p+1, 2q+1)N(a,b) -$$

$$-\beta(p+2, q+2)c(a-b)^2 S(a,b) + \frac{c^2}{30}(a-b)^4,$$

which is the required result.

Now we discuss special cases.

I. If $f = g$ in Theorem 2.2, then it reduces to the following result.

Corollary 2.1. Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be strongly convex functions with modulus $c > 0$. If $f \in L[a,b]$, then

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)\mathrm{d}x \le$$
$$\le \beta(p+q+1, p+q+1)\left[f^2(a)+f^2(b)\right] + 2\beta(2p+1, 2q+1)\left[f(a)f(b)\right] -$$
$$-2\beta(p+2, q+2)c(a-b)^2[f(a)+f(b)] + \frac{c^2}{30}(a-b)^4.$$

II. If $g(a+b-x) = g(x)$ in Theorem 2.2, then it reduces to the following result.

Corollary 2.2. Let $f, g : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be strongly beta convex functions with modulus $c > 0$. If $fg \in L[a,b]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \leq$$
$$\leq \beta(2p+1, 2q+1)M(a,b) + \beta(p+q+1, p+q+1)N(a,b) -$$
$$-\beta(p+2, q+2)c(a-b)^2 S(a,b) + \frac{c^2}{30}(a-b)^4,$$

where $M(a,b)$, $N(a,b)$ and $S(a,b)$ are given by (4), (5) and (6) respectively.

Next, we prove Hermite-Hadamard-Fejer type inequality for strongly beta-convex functions. To prove this, we need following result.

Lemma 2.2. Let $f : I = [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be strongly beta-convex function. Then

$$f(a+b-x) \leq [(1-t)^p t^q + t^p (1-t)^q][f(a) + f(b)] - f(x)$$
$$-2ct(1-t)(a-b)^2.$$

Proof. As we know that $x \in [a,b]$, can be represented as $x = (1-t)a + tb$, for all $t \in [0,1]$.

Thus

$$f(a+b-x) = f(ta + (1-t)b) \leq$$
$$\leq t^p (1-t)^q f(a) + (1-t)^p t^q f(b) - ct(1-t)(a-b)^2 =$$
$$= (1-t)^p t^q [f(a) + f(b)] + t^p (1-t)^q [f(a) + f(b)] - ct(1-t)(a-b)^2 -$$
$$-[(1-t)^p t^q f(a) + t^p (1-t)^q f(b) - ct(1-t)(a-b)^2] - ct(1-t)(a-b)^2 \leq$$
$$\leq [(1-t)^p t^q + t^p (1-t)^q][f(a) + f(b)] - f(x) - 2ct(1-t)(a-b)^2,$$

which is the required result.

Theorem 2.3. Let $f : I = [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be strongly beta-convex function. If $f \in L[a,b]$, then

$$2^{p+q-1} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\mathrm{d}x + \frac{2^{p+q}c}{8}\int_a^b (a+b-2x)^2 g(x)\mathrm{d}x \leq$$
$$\leq \int_a^b f(x)g(x)\mathrm{d}x \leq$$
$$\leq \frac{f(a)+f(b)}{2}\int_a^b \left[\left(\frac{b-x}{b-a}\right)^p \left(\frac{x-a}{b-a}\right)^q + \left(\frac{x-a}{b-a}\right)^p \left(\frac{b-x}{b-a}\right)^q\right] g(x)\mathrm{d}x -$$
$$-c(a-b)^2 \int_a^b \left(\frac{x-a}{b-a}\right)\left(\frac{b-x}{b-a}\right) g(x)\mathrm{d}x,$$

where $g:[a,b]\subseteq\mathbb{R}\setminus\{0\}$ is symmetric, nonnegative, integrable and satisfies

$$g(x)=g(a+b-x),\quad \forall x\in[a,b].$$

Proof. Using the given fact and Lemma 2.2, we have

$$2^{p+q-1}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\mathrm{d}x+\frac{2^{p+q}c}{8}\int_a^b(a+b-2x)^2g(x)\mathrm{d}x=$$

$$=2^{p+q-1}\int_a^b f\left(\frac{a+b-x+x}{2}\right)g(x)\mathrm{d}x+\frac{2^{p+q}c}{8}\int_a^b(a+b-2x)^2g(x)\mathrm{d}x\le$$

$$\le 2^{p+q-1}\int_a^b\left[\frac{1}{2^{p+q}}(f(a+b-x)+f(x))-\frac{c}{4}(a+b-2x)^2\right]g(x)\mathrm{d}x+$$

$$+\frac{2^{p+q}c}{8}\int_a^b(a+b-2x)^2g(x)\mathrm{d}x=$$

$$=\frac{1}{2}\int_a^b f(a+b-x)g(x)\mathrm{d}x+\frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x=$$

$$=\int_a^b f(x)g(x)\mathrm{d}x=$$

$$=\frac{1}{2}\int_a^b f(a+b-x)g(x)\mathrm{d}x+\frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x\le$$

$$\le\frac{1}{2}\int_a^b[[(1-t)^pt^q+t^p(1-t)^q][f(a)+f(b)]-f(x)-$$

$$-2ct(1-t)(a-b)^2]g(x)\mathrm{d}x+\frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x=$$

$$=\frac{f(a)+f(b)}{2}[(1-t)^pt^q+t^p(1-t)^q]\int_a^b g(x)\mathrm{d}x-$$

$$-ct(1-t)(a-b)^2\int_a^b g(x)\mathrm{d}x\le$$

$$\le\frac{f(a)+f(b)}{2}\int_a^b\left[\left(\frac{b-x}{b-a}\right)^p\left(\frac{x-a}{b-a}\right)^q+\left(\frac{x-a}{b-a}\right)^p\left(\frac{b-x}{b-a}\right)^q\right]g(x)\mathrm{d}x-$$

$$-c(a-b)^2\int_a^b\left(\frac{x-a}{b-a}\right)\left(\frac{b-x}{b-a}\right)g(x)\mathrm{d}x,$$

this completes the proof.

In Theorem 2.3, if $g(x)=1$, then it reduces to the Theorem 2.1.

We need the following Lemma in order to obtain new integral inequalities related to strongly beta-convex function.

Lemma 2.3. [9]. If $f: I=[a,b]\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ is a function such that $f\in L[a,b],$ then the following equality holds for some fixed $\alpha,\eta>0.$

$$\int_a^b (x-a)^{\alpha}(b-x)^{\eta} f(x)\mathrm{d}x=(b-a)^{\alpha+\eta+1}\int_0^1 t^{\alpha}(1-t)^{\eta} f((1-t)a+tb)\mathrm{d}t.$$

Theorem 2.4. Let $f: I=[a,b]\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on the interior I° of I. If $f\in L[a,b]$ and $|f|^{\lambda}$ is strongly beta-convex function on $[a,b]$ and $\alpha,\eta>0$, $\lambda\geq 1$, then

$$\left|\int_a^b (x-a)^{\alpha}(b-x)^{\eta} f(x)\mathrm{d}x\right|\leq (b-a)^{\alpha+\eta+1}[\beta(\alpha+1,\eta+1)]^{1-\frac{1}{\lambda}}\times$$

$$\times\begin{bmatrix}(|f(a)|+|f(b)|)\beta(\alpha+q+1,\eta+p+1)\\ -c(a-b)^2\beta(\alpha+2,\eta+2)\end{bmatrix}^{\frac{1}{\lambda}}$$

Proof. Using Lemma 2.3, beta-convexity of $|f|^{\lambda}$ and power mean inequality, we have

$$\left|\int_a^b (x-a)^{\alpha}(b-x)^{\eta} f(x)\mathrm{d}x\right|=$$

$$=(b-a)^{\alpha+\eta+1}\int_0^1 t^{\alpha}(1-t)^{\eta}\,|f((1-t)a+tb)|\,\mathrm{d}t\leq$$

$$\leq (b-a)^{\alpha+\eta+1}\left(\int_0^1 t^{\alpha}(1-t)^{\eta}\mathrm{d}t\right)^{1-\frac{1}{\lambda}}\left(\int_0^1 t^{\alpha}(1-t)^{\eta}\,|f((1-t)a+tb)|^{\lambda}\,\mathrm{d}t\right)^{\frac{1}{\lambda}}\leq$$

$$\leq (b-a)^{\alpha+\eta+1}\left(\int_0^1 t^{\alpha}(1-t)^{\eta}\mathrm{d}t\right)^{1-\frac{1}{\lambda}}\left(\int_0^1 t^{\alpha}(1-t)^{\eta}\left\{\begin{matrix}(1-t)^p t^q\,|f(a)|^{\lambda}\\ +t^p(1-t)^q\,|f(b)|^{\lambda}\\ -ct(1-t)(a-b)^2\end{matrix}\right\}\mathrm{d}t\right)^{\frac{1}{\lambda}}=$$

$$=(b-a)^{\alpha+\eta+1}\left(\int_0^1 t^{\alpha}(1-t)^{\eta}\mathrm{d}t\right)^{1-\frac{1}{\lambda}}\begin{pmatrix}|f(a)|^{\lambda}\int_0^1 t^{\alpha+q}(1-t)^{\eta+p}\mathrm{d}t\\ +|f(b)|^{\lambda}\int_0^1 t^{\alpha+p}(1-t)^{\eta+q}\mathrm{d}t\\ -c(a-b)^2\int_0^1 t^{\alpha+1}(1-t)^{\eta+1}\mathrm{d}t\end{pmatrix}^{\frac{1}{\lambda}}=$$

$$= (b-a)^{\alpha+\eta+1}[\beta(\alpha+1,\eta+1)]^{1-\frac{1}{\lambda}}$$

$$\times\begin{bmatrix}(|f(a)|+|f(b)|)\beta(\alpha+q+1,\eta+p+1)\\ -c(a-b)^2\beta(\alpha+2,\eta+2)\end{bmatrix}^{\frac{1}{\lambda}}$$

the the required result.

Under the assumptions of Theorem 2.4 with $\lambda = 1$, we have the following result.

Corollary 2.3. Let $f : I = [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a,b]$ and $|f|$ is strongly beta-convex function on $[a,b]$ and $\alpha, \eta > 0$, then

$$\left|\int_a^b (x-a)^\alpha (b-x)^\eta f(x)\mathrm{d}x\right|$$

$$\le (b-a)^{\alpha+\eta+1}\begin{bmatrix}(|f(a)|+|f(b)|)\beta(\alpha+q+1,\eta+p+1)\\ -c(a-b)^2\beta(\alpha+2,\eta+2)\end{bmatrix}$$

Theorem 2.5. Let $f : I = [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a,b]$ and $|f|^\lambda$ is strongly beta-convex function on $[a,b]$ and $\alpha, \eta > 0$, then

$$\left|\int_a^b (x-a)^\alpha (b-x)^\eta f(x)\mathrm{d}x\right|$$

$$\le (b-a)^{\alpha+\eta+1}[\beta(\alpha\mu+1,\eta\mu+1)]^{\frac{1}{\mu}}$$

$$\times\left[(|f(a)|^\lambda+|f(b)|^\lambda)\beta(p+1,q+1)-\frac{c}{6}(a-b)^2\right]^{\frac{1}{\lambda}},$$

where $\dfrac{1}{\lambda}+\dfrac{1}{\mu}=1$.

Proof. Using Lemma 2.3, strongly beta-convexity of $|f|^\lambda$ and the Holder's integral inequality, we have

$$\left|\int_a^b (x-a)^\alpha (b-x)^\eta f(x)\mathrm{d}x\right| =$$

$$= (b-a)^{\alpha+\eta+1}\int_0^1 t^\alpha (1-t)^\eta\,|f((1-t)a+tb)|\,\mathrm{d}t \le$$

$$\leq (b-a)^{\alpha+\eta+1}\left(\int_0^1 t^{\alpha\mu}(1-t)^{\eta\mu}\,\mathrm{d}t\right)^{\frac{1}{\mu}}\left(\int_0^1 | f((1-t)a+tb)|^{\lambda}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \leq$$

$$\leq (b-a)^{\alpha+\eta+1}\left(\int_0^1 t^{\alpha\mu}(1-t)^{\eta\mu}\,\mathrm{d}t\right)^{\frac{1}{\mu}} \times$$

$$\times\left(\int_0^1 [(1-t)^p t^q\, | f(a)|^{\lambda} + t^p(1-t)^q\, | f(b)|^{\lambda} - ct(1-t)(a-b)^2]\mathrm{d}t\right)^{\frac{1}{\lambda}} =$$

$$= (b-a)^{\alpha+\eta+1}\left[\beta(\alpha\mu+1,\eta\mu+1)\right]^{\frac{1}{\mu}} \times$$

$$\times\left[(| f(a)|^{\lambda} + | f(b)|^{\lambda})\beta(p+1,q+1) - \frac{c}{6}(a-b)^2 \right]^{\frac{1}{\lambda}}.$$

This completes the proof.

3. Conclusion

In this paper, we have introduced a new concept of the strongly convex functions with modulus with respect to three non-negative arbitrary functions. This new concept includes the class of strongly beta-convex functions as special case. Some new Hermite-Hadamard type inequalities are derived for the strongly beta-convex functions, which are our main results.Some special cases are discussed. It is an interesting problem to derive the Hermite-Hadamard type inequalities for the newly introduced class of convex functions. These results can be extended for other classes of convex functions and related optimization problems, see, for example, [15-18] and the references therein.. The ideas and techniques of this paper may be staring point for future research work in this dynamic filed.

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Составные неравенства через сильно выпуклые бета- функции

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РЕЗЮМЕ

Мы вводим и изучаем новый класс выпуклых функций, который вызван сильно выпуклые бета-функции.Мы получаем некоторые новые неравенства Эрмита-Адамара. Также рассматриваются некоторые особые случаи. Методы этой статьи могут быть отправной точкой для дальнейшего исследования в этой динамической области.

Ключевые слова: Сильно выпуклые бета- функции, бета-выпуклые функции, неравенство Эрмита-Адамара.